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ON THE DISCRETE SPECTRUM OF NEW EXACTLY SOLVABLE QUANTUM N-BODY PROBLEM ON A LINE

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It is shown that the discrete spectrum of a quantum N-body problem with the Hamiltonian

$$H = \sum_{j=1}^{N} \left(\frac{p_j^2}{2} + 2A^2(e^{4x_j} - 2e^{2x_j})\right) + \sum_{j>k}^{N} a(a-1)\mathrm{sh}^{-2}(x_j - x_k)$$

(the Sutherland system in the Morse potential) is determined by an algebraic equation. The eigenvalues and eigenfunctions, corresponding to simple excitations of the systems, are found explicitly.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

О дискретном спектре новой точно решаемой квантовой проблемы N частиц на прямой

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Показано, что задача о нахождении дискретного спектра квантовой N-частичной проблемы с гамильтонианом

$$H = \sum_{j=1}^{N} \left(\frac{p_j^2}{2} + 2A^2(e^{4x_j} - 2e^{2x_j})\right) + \sum_{j>k}^{N} a(a-1)\sinh^{-2}(x_j - x_k)$$

/система Сазерленда в потенциале Морса/ сводится к решению алгебраического уравнения. Найдены собственные значения гамильтониана и волновые функции, соответствующие простейшим возбуждениям рассматриваемой N-частичной системы.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Till now only one exact solution was found for the problem of quantum particles on a line interacting with an external field [1-3]. This solution was obtained by Calogero [1] for the system described by the Schrödinger equation

$$\left\{\sum_{j=1}^{N} \left(\frac{p_j^2}{2} + W(x_j)\right) + \sum_{j>k}^{N} V(x_j - x_k)\right\}\psi = E\psi \qquad (1)$$

at

$$V(\xi) = a(a+1)\xi^{-2}, \quad W(\xi) = \beta\xi^{2}. \tag{2}$$

Here p_j and x_j are momenta and coordinates of particles (bosons or fermions). The spectrum of this problem is completely discrete and equidistant, i.e., differs from the spectrum of trivial N noninteracting harmonic oscillators only by an additive constant and multiplicity of the energy levels.

N-particle systems with the Hamiltonian (1) are limiting cases of more general quantum systems with N degress of freedom related to semisimple Lie algebras. These systems were studied by Olshatensky and Perelomov [3]. In a series of particular cases they found the connection between these Hamiltonians and Laplace-Beltrami operators on the symmetric spaces. In particular, it has been shown that the ground-state wave function has a factorization property and can be constructed in an explicit form [4].

In this paper we study the discrete spectrum of the quantum problem with Hamiltonian (1) at

$$V(\xi) = a(a-1)\sinh^{-2}(\xi), \quad W(\xi) = 2A^{2}(e^{4\xi} - 2e^{2\xi}). \tag{3}$$

This is the so-called Sutherland system [5] in a Morse oscillator. For the Hamiltonian (1) to be self-adjoint it is necessary to constrain constants a and A in (3): $\mathrm{Im}A = \mathrm{Im}a = 0$, $\mathrm{Re}a \geq 3/2$ [6].

The system (3), evidently, has also the states with continuous spectrum corresponding to the scattering of M particles in bound states of $(N-M)$ones, $N > M$. These processes are not discussed here; we are interested only in the non-trivial discrete spectrum of system (3).

The ground-state wave function for the Schrödinger equation (1,3) was found by us earlier in paper [4]. Note that variables in that equation are not separable even in the simplest case $N = 2$. One can also construct nontrivial quantum integrals of motion, the operators containing higher degrees of momenta commuting with each other and with the Hamiltonian.

Performing in (1,3) the change of variables $z_j = e^{2x_j}$ $(0 \leq z_j < \infty)$, we transform this equation as follows

$$\{\sum_{j=1}^{N}[-2(z_j^2\frac{\partial^2}{\partial z_j^2} + z_j\frac{\partial}{\partial z_j}) + 2A^2(z_j^2 - 2z_j)] + \sum_{j>k}^{N}\frac{4a(a-1)z_jz_k}{(z_j-z_k)^2}\}\psi = E\psi.$$

Further, for brevity we suppose the particles to obey Bose statistics. For the reduction of powers of singularities in (4) let us use a standard trick and introduce a function $\phi(z_1,\dots,z_N)$ by the relation

$$\psi(z_1,\dots,z_N) = \left(\prod_{j>k}^{N} |z_j - z_k|^{\gamma} \prod_{s=1}^{N} z_s^{\beta} \exp(-r z_s)\right) \phi(z_1,\dots,z_N). \quad (5)$$

Let us choose the constants γ and r equal to a and $A (A>0)$, respectively. The leading singularities in (4) are cancelled out and the equation for function $\phi(z_1,\dots,z_N)$ can be represented in the form

$$-2\sum_{j=1}^{N} \left[z_j^2 \frac{\partial^2\phi}{\partial z_j^2} + \left(z_j(1+2\beta) - 2A z_j^2 + 2a \sum_{j\neq k} \frac{z_j^2}{z_j - z_k}\right) \frac{\partial\phi}{\partial z_j}\right] +$$
$$+ \left[p \sum_{j=1}^{N} z_j - (E - \mathcal{E}(p))\right]\phi = 0, \quad (6)$$

where

$$p(\beta) = 2A(2a(N-1) + 1 + 2\beta - 2A),$$
$$\mathcal{E}(\beta) = -N[2\beta^2 + 2a\beta(N-1) + \frac{a^2}{3}(N-1)(2N-1)]. \quad (7)$$

According to Bose statistics $\phi(z_1,\dots,z_N)$ must be symmetric under the interchange of each two arguments. Let us represent it as a function of N symmetric polynomials $a_1,\dots,a_N$

$$\phi(z_1,\dots,z_N) \equiv \phi(a_1,\dots,a_N),$$

$$a_1 = \prod_{j=1}^{N} z_j, \quad a_\ell = \frac{\hat{B}^{\ell-1}}{(\ell-1)!} a_1, \quad \ell = 1,\dots,N, \qquad \hat{B} = \sum_{j=1}^{N} \frac{\partial}{\partial z_j}. \quad (8)$$

In what follows, where the indices of quantities $\{a_s\}$ exceed N or are less than 1, one must put $a_{N+1} = 1$, $a_0 = a_{N+2} = a_{N+3} = \dots = 0$.

By simple calculations taking into account the properties of polynomial a_1 and operator $\hat{B}$, we obtain the following relations

$$\sum_{j=1}^{N} z_j \frac{\partial a_\ell}{\partial z_j} = (N - \ell + 1) a_\ell, \quad (9a)$$

$$\sum_{j=1}^{N} z_j^2 \frac{\partial a_\ell}{\partial z_j} = a_N a_\ell + a_{\ell-1}(\ell - N - 2), \quad (9b)$$

$$\sum_{j>k}^{N} \frac{z_j^2}{z_j - z_k} \frac{\partial a_\ell}{\partial z_j} = \frac{(N-\ell+1)(N+\ell-2)}{2} a_\ell , \tag{9c}$$

$$\sum_{j=1}^{N} z_j^2 \frac{\partial a_\ell}{\partial z_j} \frac{\partial a_m}{\partial z_j} = (N-\ell+1) a_\ell a_m - \sum_{r=1}^{\ell} (\ell + m - 2r) a_r a_{\ell+m-r} . \tag{9d}$$

By (9a-d) it is easy to find the equation for function $\phi(a_1, \dots, a_N)$ (8):

$$(\hat{H}_1 + \hat{H}_2)\phi(a_1, \dots, a_N) = (E - \mathcal{E}(\beta))\phi(a_1, \dots, a_N) , \tag{10}$$

where

$$\hat{H}_1 = -2\Big\{ \sum_{\ell,m=1}^{N} [(N-\ell+1)a_\ell a_m - \sum (\ell+m-2r) a_r a_{\ell+m-r}] \frac{\partial^2}{\partial a_\ell \partial a_m} + \tag{11}$$

$$+ \sum_{\ell=1}^{N} [(N-\ell+1)(1+2\beta+a(N+\ell-2)) a_\ell \frac{\partial}{\partial a_\ell} + 2A(N-\ell+2) a_{\ell-1} \frac{\partial}{\partial a_\ell}] \Big\},$$

$$\hat{H}_2 = a_N (4A \sum_{\ell=1}^{N} a_\ell \frac{\partial}{\partial a_\ell} + p(\beta)) . \tag{12}$$

Let us choose the solutions of eq.(10) as polynomials in variables $\{a_\ell\}$ of the form

$$\phi^{(n)}(a_1, \dots, a_N) = \sum_{\nu=0}^{n} \sum_{\substack{j_1+\dots+j_N=\nu \\ 0 \le j \le \nu}} c^{(\nu)}_{j_1 \dots j_N} a_1^{j_1} \dots a_N^{j_N} . \tag{13}$$

Evidently the operator $\hat{H}_1$ does not raise the maximal degree of these polynomials. As for $\hat{H}_2$, it adds 1 to this degree if the condition

$$p(\beta) + 4An = 0 \tag{14}$$

is not satisfied.

If $p(\beta)$ obeys (14), the sum $\hat{H}_1 + \hat{H}_2$ is a linear operator acting in the space of polynomials in N variables with degree not exceeding n. The dimensionality of this space is $\frac{(N+n)!}{n!N!}$. The eigenvalues of this operator represent (at given n) the spectrum of the considered problem up to a constant $\mathcal{E}(\beta)$. In this case the parameter β and constant $\mathcal{E}(\beta)$ are completely determined by the integer n. The normalizability condition for the wave function (5) has the form

$$\int dz_1 \dots dz_N \prod_{j>k}^{N} |z_j - z_k|^{2a} \prod_{s=1}^{N} z_s^{2\beta-1} \exp(-2Az_s) |\phi(z_1, \dots, z_N)|^2 < \infty . \tag{15}$$

From (13) and (15) it follows that $\beta(n)$ must obey the condition $\beta > 0$, i.e., the integer n is restricted:

$$A - \alpha(N-1) > n + 1/2. \qquad (16)$$

So, we formulate the way for constructing the discrete spectrum of the Hamiltonian (1,3): for the integers n which obey the condition (16) one must find the matrix of the operator $\hat{H}_1 + \hat{H}_2$ according to (10-12) in the basis $\{a_1^{j_1} \dots a_N^{j_N}\}$, $j_1 + \dots + j_N \le n$. The eigenvalues of this matrix, up to constant $\mathcal{E}(\beta_n)$, represent the spectrum to be found. So, the problem is reduced to an algebraic equation. This problem is apparently simple in the case $n=1$, where $\phi(a_1, \dots, a_N)$ has the form

$$\phi(a_1, \dots, a_N) = \sum_{\ell=1}^{N} c_\ell a_\ell + c_{\ell+1}. \qquad (17)$$

The operator $\partial^2/\partial a_\ell \partial a_m$ acting on the function (17) reduces it to zero. The matrix of the operator $\hat{H}_1 + \hat{H}_2$ is the upper triangular one and its spectrum is determined by the diagonal elements:

$$-E_\ell^{(1)} = \frac{N}{2}\left[(2A-\alpha(N-1)-3)^2 + \frac{\alpha^2(N^2-1)}{3}\right] + 2(N-\ell+1)(2A-2-\alpha(N-\ell)),$$
$$\ell = 2, \dots, N+1. \qquad (18)$$

Note that at $3/2 < A - \alpha(N-1) < 5/2$ the values $E_\ell^{(1)}$ (18) and the ground-state energy /4/ represent the whole discrete spectrum of the problem we are interested in. One can also (up to normalization constants) determine the wave functions corresponding to eigenvalues $E_\ell^{(1)}$:

$$\phi_\ell^{(1)}(a_1, \dots, a_N) = \mathcal{A}_\ell^{(1)} \sum_{j=1}^{\ell} (-1)^{\ell-j} \left(\frac{2A}{\alpha}\right)^{\ell-j} \frac{(N-j+1)!}{(\ell-j)!(N-\ell+1)!} \times$$
$$\times \frac{\Gamma\left(\frac{2A-2}{\alpha} - 2N + j + \ell - 1\right)}{\Gamma\left(\frac{2A-2}{\alpha} - 2(N-\ell) - 1\right)} a_j.$$

When $n > 1$ it is easy to see that the matrix of the operator $\hat{H}_1$ in the basis $\{a_1^{j_1} \dots a_N^{j_N}\}$ is no longer upper triangular, and the determination of eigenvalues represents a much more complicated problem. Possibly, for the solution of this problem one can apply algebraic methods as it is made in papers [2].

We can, however, immediately find one of these eigenvalues. Really, note that $\hat{H}_1 + \hat{H}_2$ transforms (under the condition (14)) the linear space of polynomials of the form (13) to a space of a lower dimensionality: its action on $a_1^{j_1} \dots a_N^{j_N}$ does not contain the polynomial of zero degree. So, there exists a subspace corresponding to the zero eigenvalue of this operator and $E^{(n)} = \mathcal{E}(\beta_n) = -\frac{N}{2}[\frac{a^2}{3}(N^2-1) + (2A - a(N-1) - 2n - 1)^2]$ are eigenvalues of the Hamiltonian. It is evident that the set $\{E^{(n)}\}$, up to a constant $-\frac{Na^2}{6}(N^2-1)$, coincides with that set of the levels of unperturbed system of N particles in the Morse oscillator, for which each particle is on an n-th level and the constant A is "renormalized" by the interaction: $A \to A - \frac{a}{2}(N-1)$.

Detailed calculations and investigation of the whole discrete spectrum for the case $n \geq 2$ will be pulblished elsewhere.

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